Homework 1

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Question 1

A. Convert the following numbers to their decimal representation. Show your work.

1. $10011011_2 = \mathbf{155}$

   Solution.

$$\begin{aligned} 10011011_2 &= 1*2^7 + 1*2^4 + 1*2^3 + 1*2^1 + 1*2^0 \\ &= 128 + 16 + 8 + 2 + 1 \\ &= 155 \end{aligned}$$

□

2. $456_7 = \mathbf{237}$

   Solution.

$$\begin{aligned} 456_7 &= 4*7^2 + 5*7^1 + 6*7^0 \\ &= 196 + 35 + 6 \\ &= 237 \end{aligned}$$

□

3. $38A_{16} = \mathbf{906}$

   Solution.

$$\begin{aligned} 38A_{16} &= 3*16^2 + 8*16^1 + 10*16^0 \\ &= 768 + 128 + 10 \\ &= 906 \end{aligned}$$

□

4. $2214_5 = \mathbf{309}$

Solution.

$$\begin{aligned} 2214_5 &= 2 * 5^3 + 2 * 5^2 + 1 * 5^1 + 4 * 5^0 \\ &= 250 + 50 + 5 + 4 \\ &= 309 \end{aligned}$$

□

B. Convert the following numbers to their binary representation:

1. $69_{10} = \mathbf{1000101_2}$

Solution. Divide 69 by 2 and then recursively use quotients as dividends until the quotient is zero. Keep all the remainders in reverse order.

Dividend	Divisor	Quotient	Remainder	
69		34	**1**	
34		17	**0**	
17		8	**1**	$\uparrow$
8	2	4	**0**	$\mathbf{1000101_2}$
4		2	**0**	
2		1	**0**	
1		0	**1**	

Thus the binary representation of 69_{10} is $\mathbf{1000101_2}$.

□

2. $485_{10} = \mathbf{111100101_2}$

Solution. Divide 485 by 2 and then recursively use quotients as dividends until the quotient is zero. Keep all the remainders in reverse order.

Dividend	Divisor	Quotient	Remainder	
485		242	**1**	
242		121	**0**	
121		60	**1**	
60		30	**0**	
30	2	15	**0**	$\uparrow$ $\mathbf{111100101_2}$
15		7	**1**	
7		3	**1**	
3		1	**1**	
1		0	**1**	

Thus the binary representation of 485_{10} is $\mathbf{111100101_2}$. □

3. $6D1A_{16} = \mathbf{110110100011010_2}$

   Solution. Since $2^4 = 16$, each digit of a hexadecimal number can be represented by a four-digit binary number. Then each digit of $6D1A_{16}$ can be represented as below:

hexadecimal	6_{16}	D_{16}	1_{16}	A_{16}
binary	0110_2	1101_2	0001_2	1010_2

   Thus the binary representation of $6D1A_{16}$ is $\mathbf{110110100011010_2}$. □

C. Convert the following numbers to their hexadecimal representation:

1. $1101011_2 = \mathbf{6B_{16}}$

   Solution. From right to left, taking four digits as a group, the binary number 1101011_2 can be divided into two groups.

   $$0110_2 = 6_{16}$$
   $$1011_2 = B_{16}$$

   Thus the hexadecimal representation of 1101011_2 is $\mathbf{6B_{16}}$. □

2. $895_{10} = \mathbf{37F_{16}}$

Solution. Divide 895_{10} by 16_{10} and then recursively use quotients as dividends until the quotient is zero. Keep all the remainders in reverse order.

$$
\begin{aligned}
895/16 &= 55......\mathbf{F_{16}}\\
55/16 &= 3......\mathbf{7_{16}}\\
3/16 &= 0......\mathbf{3_{16}}
\end{aligned}
$$

Dividend	Divisor	Quotient	Remainder	
895_{10}		55_{10}	$\mathbf{F_{16}}$	
55_{10}	16_{10}	3_{10}	$\mathbf{7_{16}}$	$\uparrow$ $\mathbf{37F_{16}}$
3_{10}		0_{10}	$\mathbf{3_{16}}$	

Thus the hexadecimal representation of 895_{10} is $\mathbf{37F_{16}}$.

$\square$

Question 2

Solve the following, do all calculation in the given base. Show your work.

1. $7566_8 + 4515_8 = \mathbf{14303_8}$

```
     1 1 1
     7 5 6 6
   + 4 5 1 5
 ------------
   1 4 3 0 3
```

2. $10110011_2 + 1101_2 = \mathbf{11000000_2}$

```
       1 1 1 1 1 1
     1 0 1 1 0 0 1 1
   +         1 1 0 1
   ------------------
     1 1 0 0 0 0 0 0
```

3. $7A66_{16} + 45C5_{16} = \mathbf{C02B_{16}}$

```
     1 1
     7 A 6 6
  +  4 5 C 5
  -----------
     C 0 2 B
```

4. $3022_5 - 2433_5 = \mathbf{34_5}$

```
     2 4 1
     3 0 2 2
  -  2 4 3 3
  -----------
         3 4
```

Question 3

A. Convert the following numbers to their 8-bits two's complement representation. Show your work.

1. $124_{10} = \mathbf{01111100}$

   Solution. Since 124_{10} is a positive number, its 8-bits two's complement is **01111100**.

Dividend	Divisor	Quotient	Remainder
124		62	**0**
62		31	**0**
31		15	**1**
15	2	7	**1**
7		3	**1**
3		1	**1**
1		0	**1**

$\uparrow$ $\mathbf{1111100_2}$ $\square$

2. $-124_{10} = \mathbf{10000100}$

   Solution. Since -124_{10} is a negative number, and from above, we know that $124_{10} = 1111100_2$, so 8-bits two's complement of -124_{10} equals to $100000000_2 - 1111100_2$.

```
   1 1 1 1 1 1
   1 0 0 0 0 0 0 0 0
-      1 1 1 1 1 0 0
---------------------
     1 0 0 0 0 1 0 0
```

$\square$

3. $109_{10} = \mathbf{01101101}$

   Solution. Since 109_{10} is a positive number, its 8-bits two's complement is **01101101**.

Dividend	Divisor	Quotient	Remainder	
109		54	**1**	
54		27	**0**	
27		13	**1**	
13	2	6	**1**	$\mathbf{1101101_2}$
6		3	**0**	
3		1	**1**	
1		0	**1**	

□

4. $-79_{10} = \mathbf{10110001}$

Solution. Since -79_{10} is a negative number, its 8-bits two's complement equals to $100000000_2 - 79_{10}$.

Dividend	Divisor	Quotient	Remainder	
79		39	**1**	
39		19	**1**	
19		9	**1**	
9	2	4	**1**	$\mathbf{1001111_2}$
4		2	**0**	
2		1	**0**	
1		0	**1**	

From above, $100000000_2 - 79_{10} = 100000000_2 - 1001111_2$.

```
     1 1 1 1 1 1 1 1
     1 0 0 0 0 0 0 0 0
-        1 0 0 1 1 1 1
----------------------
       1 0 1 1 0 0 0 1
```

□

B. Convert the following numbers (represented as 8-bit two's complement) to their decimal representation. Show your work.

1. $00011110_{8\text{ bit 2's comp}} = \mathbf{30}$

Solution. Since the first digit is 0, which means it's a positive number, so it equals to 11110_2.

$$\begin{aligned} 11110_2 &= 1*2^4 + 1*2^3 + 1*2^2 + 1*2^1 \\ &= 16 + 8 + 4 + 2 \\ &= 30 \end{aligned}$$

Thus the decimal representation of $00011110_{8\text{ bit 2's comp}}$ is **30**. □

2. $11100110_{8\text{ bit 2's comp}} = \mathbf{-26}$

   Solution. Since the first digit is 1, which means it's a negative number.

$$\begin{aligned} 11100110_{8\text{ bit 2's comp}} &= -2^7 + 1100110_2 \\ &= -128 + 1*2^6 + 1*2^5 + 1*2^2 + 1*2^1 \\ &= -128 + 64 + 32 + 4 + 2 \\ &= -26 \end{aligned}$$

   Thus the decimal representation of $11100110_{8\text{ bit 2's comp}}$ is $\mathbf{-26}$. □

3. $00101101_{8\text{ bit 2's comp}} = \mathbf{45}$

   Solution. Since the first digit is 0, which means it's a positive number, so it equals to 101101_2.

$$\begin{aligned} 101101_2 &= 1*2^5 + 1*2^3 + 1*2^2 + 1*2^0 \\ &= 32 + 8 + 4 + 1 \\ &= 45 \end{aligned}$$

   Thus the decimal representation of $00101101_{8\text{ bit 2's comp}}$ is **45**. □

4. $10011110_{8\text{ bit 2's comp}} = \mathbf{-98}$

   Solution. Since the first digit is 1, which means it's a negative number.

$$\begin{aligned} 10011110_{8\text{ bit 2's comp}} &= -2^7 + 11110_2 \\ &= -128 + 1*2^4 + 1*2^3 + 1*2^2 + 1*2^1 \\ &= -128 + 16 + 8 + 4 + 2 \\ &= -98 \end{aligned}$$

   Thus the decimal representation of $10011110_{8\text{ bit 2's comp}}$ is $\mathbf{-98}$. □

Question 4

1. Write a truth table for each expression.

(a) $\neg(p \lor q)$

p	q	$p \lor q$	$\neg(\mathbf{p} \lor \mathbf{q})$
T	T	T	**F**
T	F	T	**F**
F	T	T	**F**
F	F	F	**T**

(b) $r \lor (p \land \neg q)$

r	p	q	$\neg q$	$p \land \neg q$	$\mathbf{r} \lor (\mathbf{p} \land \neg\mathbf{q})$
T	T	T	F	F	**T**
T	T	F	T	T	**T**
T	F	T	F	F	**T**
T	F	F	T	F	**T**
F	T	T	F	F	**F**
F	T	F	T	T	**T**
F	F	T	F	F	**F**
F	F	F	T	F	**F**

2. Give a truth table for each expression.

(a) $(p \to q) \to (q \to p)$

p	q	$p \to q$	$q \to p$	$(\mathbf{p} \to \mathbf{q}) \to (\mathbf{q} \to \mathbf{p})$
T	T	T	T	**T**
T	F	F	T	**T**
F	T	T	F	**F**
F	F	T	T	**T**

(b) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

p	q	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(\mathbf{p} \leftrightarrow \mathbf{q}) \oplus (\mathbf{p} \leftrightarrow \neg\mathbf{q})$
T	T	F	T	F	**T**
T	F	T	F	T	**T**
F	T	F	F	T	**T**
F	F	T	T	F	**T**

Question 5

1. Consider the following pieces of identification a person might have in order to apply for a credit card:

   B: Applicant presents a birth certificate.

   D: Applicant presents a driver's license.

   M: Applicant presents a marriage license.

   Write a logical expression for the requirements under the following conditions:

   (a) The applicant must present at least two of the following forms of identification: birth certificate, driver's license, marriage license.

   Solution. $(B \wedge D) \vee (B \wedge M) \vee (D \wedge M)$ □

   (b) Applicant must present either a birth certificate or both a driver's license and a marriage license.

   Solution. $B \vee (D \wedge M)$ □

2. Define the following propositions:

   s: a person is a senior

   y: a person is at least 17 years of age

   p: a person is allowed to park in the school parking lot

   Express each of the following English sentences with a logical expression:

   (a) A person can park in the school parking lot if they are a senior or at least seventeen years of age.

   Solution. $(s \vee y) \rightarrow p$ □

   (b) Being 17 years of age is a necessary condition for being able to park in the school parking lot.

   Solution. $p \rightarrow y$ □

   (c) A person can park in the school parking lot if and only if the person is a senior and at least 17 years of age.

   Solution. $p \leftrightarrow (s \wedge y)$ □

(d) Being able to park in the school parking lot implies that the person is either a senior or at least 17 years old.

Solution. $p \to (s \vee y)$ □

3. Use the definitions of the variables below to translate each English statement into an equivalent logical expression.

- y: the applicant is at least eighteen years old
- p: the applicant has parental permission
- c: the applicant can enroll in the course

(a) The applicant can enroll in the course only if the applicant has parental permission.

Solution. $c \to p$ □

(b) Having parental permission is a necessary condition for enrolling in the course.

Solution. $c \to p$ □

Question 6

1. Give an English sentence in the form "If...then..." that is equivalent to each sentence.

   (a) Maintaining a B average is necessary for Joe to be eligible for the honors program.

   Solution. **If Joe is eligible for the honors program, then he maintains a B average.** □

   (b) Rajiv can go on the roller coaster only if he is at least four feet tall.

   Solution. **If Rajiv can go on the roller coaster, then he is at least four feet tall.** □

   (c) Rajiv can go on the roller coaster if he is at least four feet tall.

   Solution. **If Rajiv is at least four feet tall, then he can go on the roller coaster.** □

2. The variable p is true, q is false, and the truth value for variable r is unknown. Indicate whether the truth value of each logical expression is true, false, or unknown.

   (a) $(p \vee r) \leftrightarrow (q \wedge r)$

   Solution. **False.**
   Since p is true, $p \vee r$ is true.
   Since q is false. $q \wedge r$ is false.
   Thus, $(p \vee r) \leftrightarrow (q \wedge r)$ is false.

   □

   (b) $(p \wedge r) \leftrightarrow (q \wedge r)$

   Solution. **Unknown.**
   Since p is true and r is unknown, $p \wedge r$ is unknown.
   Since q is false and r is unknown, $q \wedge r$ is false.
   Thus, $(p \wedge r) \leftrightarrow (q \wedge r)$ is unknown.

   □

   (c) $p \rightarrow (r \vee q)$

Solution. **Unknown.**
Since q is false and r is unknown, $r \lor q$ is unknown.
Since p is true and $r \lor q$ is unknown, $p \to (r \lor q)$ is unknown.

□

(d) $(p \land q) \to r$

Solution. **True.**
Since p is true and q is false, $p \land q$ is false.
Since r is unknown, if r is true, $(p \land q) \to r$ is true, if r is false, $(p \land q) \to r$ is also true.
Thus, $(p \land q) \to r$ is true.

□

Question 7

Define the following propositions:

j: Sally got the job.

l: Sally was late for her interview

r: Sally updated her resume.

Express each pair of sentences using logical expressions. Then prove whether the two expressions are logically equivalent.

(a)
- If Sally did not get the job, then she was late for her interview or did not update her resume.
- If Sally updated her resume and was not late for her interview, then she got the job.

Solution. **Logically equivalent.**

The first expression is $\neg j \to (l \lor \neg r)$.

The second expression is $(r \land \neg l) \to j$.

Below is the truth table for both expressions.

j	l	r	$\neg j \to (l \lor \neg r)$	$(r \land \neg l) \to j$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	F
F	F	F	T	T

The truth table of both expressions is the same.Thus the two expressions are logically equivalent.

□

(b)
- If Sally got the job then she was not late for her interview.
- If Sally did not get the job, then she was late for her interview.

Solution. **Not logically equivalent.**

The first expression is $j \to \neg l$.

The second expression is $\neg j \to l$.

Below is the truth table for both expressions.

j	l	$j \to \neg l$	$\neg j \to l$
T	T	F	T
T	F	T	T
F	T	T	T
F	F	T	F

The truth table of each expression is not the same. Thus the two expressions are not logically equivalent.

$\square$

(c)
- If Sally updated her resume or she was not late for her interview, then she got the job.
- If Sally got the job, then she updated her resume and was not late for her interview.

Solution. **Not logically equivalent.**

The first expression is $(r \vee \neg l) \to j$.

The second expression is $j \to (r \wedge \neg l)$.

Below is the truth table for both expressions.

j	l	r	$(r \vee \neg l) \to j$	$j \to (r \wedge \neg l)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	F	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	T

The truth table of each expression is not the same. Thus the two expressions are not logically equivalent.

$\square$

Question 8

1. Use the laws of propositional logic to prove the following:

   (a) $(p \to q) \land (p \to r) \equiv p \to (q \land r)$

   Solution.

$$\begin{aligned}(p \to q) \land (p \to r) &\equiv (\neg p \lor q) \land (\neg p \lor r)\\ &\equiv \neg p \lor (q \land r)\\ &\equiv p \to (q \land r)\end{aligned}$$

□

   (b) $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

   Solution.

$$\begin{aligned}\neg(p \lor (\neg p \land q)) &\equiv \neg((p \lor \neg p) \land (p \lor q))\\ &\equiv \neg(T \land (p \lor q))\\ &\equiv \neg(p \lor q)\\ &\equiv \neg p \land \neg q\end{aligned}$$

□

   (c) $(p \land q) \to r \equiv (p \land \neg r) \to \neg q$

   Solution.

$$\begin{aligned}(p \land q) \to r &\equiv \neg(p \land q) \lor r\\ &\equiv \neg p \lor \neg q \lor r\\ &\equiv (\neg p \lor r) \lor \neg q\\ &\equiv \neg(\neg p \lor r) \to \neg q\\ &\equiv (p \land \neg r) \to \neg q\end{aligned}$$

□

2. Use the laws of propositional logic to prove that each statement is a tautology.

   (a) $\neg r \lor (\neg r \to p)$

Solution.

$$\begin{aligned}
\neg r \vee (\neg r \to p) &\equiv \neg r \vee (\neg\neg r \vee p) \\
&\equiv \neg r \vee (r \vee p) \\
&\equiv \neg r \vee r \vee p \\
&\equiv T \vee p \\
&\equiv T
\end{aligned}$$

□

(b) $\neg(p \to q) \to \neg q$

Solution.

$$\begin{aligned}
\neg(p \to q) \to \neg q &\equiv \neg\neg(\neg p \vee q) \vee \neg q \\
&\equiv \neg p \vee q \vee \neg q \\
&\equiv \neg p \vee (q \vee \neg q) \\
&\equiv \neg p \vee T \\
&\equiv T
\end{aligned}$$

□

Question 9

1. Consider the following statements in English. Write a logical expression with the same meaning. The domain is the set of all real numbers.

   (a) There is a number that is equal to its square.

   Solution. $\exists \mathbf{x}(\mathbf{x} = \mathbf{x^2})$ □

   (b) Every number is less than or equal to its square plus 1.

   Solution. $\forall \mathbf{x}(\mathbf{x} \leq \mathbf{x^2} + \mathbf{1})$ □

2. In the following question, the domain is a set of employees who work at a company. Ingrid is one of the employees at the company. Define the following predicates:

S(x): x was sick yesterday

W(x): x went to work yesterday

V(x): x was on vacation yesterday

Translate the following English statements into a logical expression with the same meaning.

   (a) Everyone was well and went to work yesterday.

   Solution. $\forall \mathbf{x}(\neg \mathbf{S}(\mathbf{x}) \land \mathbf{W}(\mathbf{x}))$ □

   (b) Everyone who was sick yesterday did not go to work.

   Solution. $\forall \mathbf{x}(\mathbf{S}(\mathbf{x}) \to \neg \mathbf{W}(\mathbf{x}))$ □

   (c) Yesterday someone was sick and went to work.

   Solution. $\exists \mathbf{x}(\mathbf{S}(\mathbf{x}) \land \mathbf{W}(\mathbf{x}))$ □

Question 10

1. The domain for this question is the set a, b, c, d, e. The following table gives the value of predicates P, Q, and R for each element in the domain. For example, Q(c) = T because the truth value in the row labeled c and the column Q is T. Using these values, determine whether each quantified expression evaluates to true or false.

	P(x)	Q(x)	R(x)
a	T	T	F
b	T	F	F
c	F	T	F
d	T	T	F
e	T	T	T

(a) $\exists x((x = c) \to P(x))$

Solution. **True.**
When $x \neq c$, which is when $x = a$ or $x = b$ or $x = d$ or $x = e$, P(x) is true, then $(x = c) \to P(x)$ is true.
Thus, $\exists x((x = c) \to P(x))$ is true. □

(b) $\exists x(Q(x) \wedge R(x))$

Solution. **True.**
When $x = e$, Q(x) is true and R(x) is true, then $Q(x) \wedge R(x)$ is true.
Thus, $\exists x(Q(x) \wedge R(x))$ is true.
□

(c) $Q(a) \wedge P(d)$

Solution. **True.**
Since $Q(a)$ is true and $P(d)$ is true, $Q(a) \wedge P(d)$ is true.
□

(d) $\forall x((x \neq b) \to Q(x))$

Solution. **True.**
When $x \neq b$, Q(x) is true, then $\forall x((x \neq b) \to Q(x))$ is true.
□

(e) $\forall x(P(x) \vee R(x))$

Solution. **False.**
If $x = c$, $P(x)$ is false and $R(x)$ is false, thus $\forall x(P(x) \vee R(x))$ is false.

$\square$

(f) $\forall x(R(x) \to P(x))$

Solution. **True.**
If $x = c$, $R(x)$ is false and $P(x)$ is false, then $R(x) \to P(x)$ is true.
If $x \neq c$, $P(x)$ is true, then $R(x) \to P(x)$ is true.
Thus, $\forall x(R(x) \to P(x))$ is true.

$\square$

(g) $\exists x(Q(x) \vee R(x))$

Solution. **True.**
If $x = a$, $Q(x)$ is true and $R(x)$ is false, then $Q(x) \vee R(x)$ is true.
Thus, $\exists x(Q(x) \vee R(x))$ is true.

$\square$

2. The tables below show the values of predicates P(x, y), Q(x, y), and S(x, y) for every possible combination of values of the variables x and y. The row number indicates the value for x and the column number indicates the value for y. The domain for x and y is 1, 2, 3.

P	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

Q	1	2	3
1	F	F	F
2	T	T	T
3	T	F	F

S	1	2	3
1	F	F	F
2	F	F	F
3	F	F	F

Indicate whether each of the quantified statements is true or false.

(a) $\exists x \forall y Q(x, y)$

Solution. **True**
Let $x = 2$, then $Q(x, 1)$, $Q(x, 2)$ and $Q(x, 3)$ are all true. $\square$

(b) $\exists y \forall x P(x, y)$

Solution. **True**
Let $y = 1$, then $P(1, y)$, $P(2, y)$ and $P(3, y)$ are all true. $\square$

(c) $\exists x \exists y S(x, y)$

Solution. **False**
There is no pair (x, y) such that $S(x, y)$ is true. □

(d) $\forall x \exists y Q(x, y)$

Solution. **False**
Let $x = 1$, then there is no y such that Q(1, y) is true. □

(e) $\forall x \exists y P(x, y)$

Solution. **True**
When $x = 1$, let $y = 1$, $P(x, y)$ is true.
When $x = 2$, let $y = 1$, $P(x, y)$ is true.
When $x = 3$, let $y = 1$, $P(x, y)$ is true. □

(f) $\forall x \forall y P(x, y)$

Solution. **False**
Let $x = 2$ and $y = 2$, $P(x, y)$ is false. □

(g) $\exists x \exists y Q(x, y)$

Solution. **True**
Let $x = 2$ and $y = 2$, $Q(x, y)$ is true.

□

(h) $\forall x \forall y \neg S(x, y)$

Solution. **True**
For each pair (x, y), $S(x, y)$ is false.

□

Question 11

1. Translate each of the following English statements into logical expressions. The domain is the set of all real numbers.

   (a) There are two numbers whose sum is equal to their product.

   Solution. $\exists x \exists y(x + y = xy)$ □

   (b) The ratio of every two positive numbers is also positive.

   Solution. $\forall x \forall y(((x > 0) \wedge (y > 0)) \rightarrow x/y > 0)$ □

   (c) The reciprocal of every positive number less than one is greater than one.

   Solution. $\forall x(((x > 0) \wedge (x < 1)) \rightarrow 1/x > 1)$ □

   (d) There is no smallest number.

   Solution. $\forall x \exists y(x > y)$ □

   (e) Every number other than 0 has a multiplicative inverse.

   Solution. $\forall x \exists y((x \neq 0) \rightarrow xy = 1)$ □

2. The domain is a group working on a project at a company. One of the members of the group is named Sam. Define the following predicates.

   - P(x, y): x knows y's phone number. (A person may or may not know their own phone number.)
   - D(x): x missed the deadline.
   - N(x): x is a new employee.

   Give a logical expression for each of the following sentences.

   (a) There is at least one new employee who missed the deadline.

   Solution. $\exists x(N(x) \wedge D(x))$ □

   (b) Sam knows the phone number of everyone who missed the deadline.

   Solution. $\forall x(D(x) \rightarrow P(Sam, x))$ □

   (c) There is a new employee who knows everyone's phone number.

Solution. $\exists x \forall y (N(x) \wedge P(x, y))$ □

(d) Exactly one new employee missed the deadline.

Solution. $\exists x \forall y ((N(x) \wedge D(x)) \wedge (((y \neq x) \wedge N(y)) \rightarrow \neg D(y)))$ □

3. The domain for the first input variable to predicate T is a set of students at a university. The domain for the second input variable to predicate T is the set of Math classes offered at that university. The predicate T(x, y) indicates that student x has taken class y. Sam is a student at the university and Math 101 is one of the courses offered at the university. Give a logical expression for each sentence.

   (a) Every student has taken at least one class other than Math 101.

   Solution. $\forall x \exists y (T(x, y) \wedge (y \neq (Math101)))$ □

   (b) There is a student who has taken every math class other than Math 101.

   Solution. $\exists x \forall y ((y \neq (Math101)) \rightarrow T(x, y))$ □

   (c) Everyone other than Sam has taken at least two different math classes.

   Solution. $\forall x \exists y \exists z ((x \neq Sam) \rightarrow ((y \neq z) \wedge T(x, y) \wedge T(x, z)))$ □

   (d) Sam has taken exactly two math classes.

   Solution. $\exists x \exists y \forall z ((x \neq y) \wedge T(Sam, x) \wedge T(Sam, y) \wedge (((z \neq x) \wedge (z \neq y)) \rightarrow \neg T(Sam, z)))$ □

Question 12

1. In the following question, the domain is a set of male patients in a clinical study. Define the following predicates:

P(x): x was given the placebo

D(x): x was given the medication

M(x): x had migraines

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

(a) Every patient was given the medication or the placebo or both.

- $\forall x(D(x) \vee P(x))$
- Negation: $\neg\forall x(D(x) \vee P(x))$
- Applying De Morgan's law: $\exists x(\neg D(x) \wedge \neg P(x))$
- English: There is a patient who was neither given the medication nor the placebo.

(b) There is a patient who took the medication and had migraines.

- $\exists x(D(x) \wedge M(x))$
- Negation: $\neg\exists x(D(x) \wedge M(x))$
- Applying De Morgan's law: $\forall x(\neg D(x) \vee \neg M(x))$
- English: Every patient either was not given the medication or did not have migraines(or both).

(c) Every patient who took the placebo had migraines. (Hint: you will need to apply the conditional identity, $p \to q \equiv \neg p \vee q$.)

- $\forall x(P(x) \to M(x))$
- Negation:$\neg\forall x(P(x) \to M(x))$
- Applying De Morgan's law: $\exists x(P(x) \wedge \neg M(x))$
- English: There is a patient who took the placebo and did not have migraines.

(d) There is a patient who had migraines and was given the placebo.

- $\exists x(M(x) \land P(x))$
- Negation: $\neg\exists x(M(x) \land P(x))$
- Applying De Morgan's law: $\forall x(\neg M(x) \lor \neg P(x))$
- English: Every patient either did not have migraines or was not given the placebo (or both).

2. Write the negation of each of the following logical expressions so that all negations immediately precede predicates. In some cases, it may be necessary to apply one or more laws of propositional logic.

(a) $\exists x\forall y(P(x,y) \to Q(x,y))$

Solution.

$$\begin{aligned}\neg\exists x\forall y(P(x,y) \to Q(x,y)) &\equiv \forall x\exists y\neg(\neg P(x,y) \lor Q(x,y))\\ &\equiv \forall x\exists y(P(x,y) \land \neg Q(x,y))\end{aligned}$$

$\square$

(b) $\exists x\forall y(P(x,y) \leftrightarrow P(y,x))$

Solution.

$$\begin{aligned}\neg\exists x\forall y(P(x,y) \leftrightarrow P(y,x)) &\equiv \forall x\exists y\neg((P(x,y) \to P(y,x)) \land (P(y,x) \to P(x,y)))\\ &\equiv \forall x\exists y\neg((\neg P(x,y) \lor P(y,z)) \land (\neg P(y,x) \lor P(x,y)))\\ &\equiv \forall x\exists y((P(x,y) \land \neg P(y,x)) \lor (P(y,x) \land \neg P(x,y)))\end{aligned}$$

$\square$

(c) $\exists x\exists yP(x,y) \land \forall x\forall yQ(x,y)$

Solution.

$$\begin{aligned}\neg(\exists x\exists yP(x,y) \land \forall x\forall yQ(x,y)) &\equiv \neg\exists x\exists yP(x,y) \lor \neg\forall x\forall yQ(x,y)\\ &\equiv \forall x\forall y\neg P(x,y) \lor \exists x\exists y\neg Q(x,y)\end{aligned}$$

$\square$