Homework 2

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Question 5

a) Solve the following questions from the Discrete Math zyBook:

1.12.2 Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

(b) $p \to (q \land r)$
$\neg q$
$\therefore \neg p$

1	$\neg q$	Hypothesis
2	$\neg q \lor \neg r$	Addition, 1
3	$\neg(\neg q \land r)$	De Morgan's law, 2
4	$p \to (q \land r)$	Hypothesis
5	$\neg p$	Modus tollens, 3, 4

(e) $p \lor q$
$\neg p \lor r$
$\neg q$
$\therefore r$

1	$p \lor q$	Hypothesis
2	$\neg p \lor r$	Hypothesis
3	$q \lor r$	Resolution, 1, 2
4	$\neg q$	Hypothesis
5	r	Disjunctive syllogism, 3, 4

1.12.3 Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

(c) One of the rules of inference is Disjunctive syllogism:

$p \lor q$
$\neg p$
$\therefore q$

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

1	$p \lor q$	Hypothesis
2	$\neg(\neg p) \lor q$	Double negation law, 1
3	$\neg p \to q$	Conditional identities, 2
4	$\neg p$	Hypothesis
5	q	Modus ponens, 3, 4

1.12.5 Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

(c) I will buy a new car and a new house only if I get a job.
I am not going to get a job.
∴ I will not buy a new car.

Solution. **Invalid**
Assume that

- c: I will buy a new car.
- h: I will buy a new house.
- j: I will get a job.

The expression can be written as below:

$(c \land h) \to j$
$\neg j$
$\therefore \neg c$

1	$(c \land h) \to j$	Hypothesis
2	$\neg j$	Hypothesis
3	$\neg(c \land h)$	Modus tollens, 1, 2
4	$\neg c \lor \neg h$	De Morgan's law

From above, we know that either $\neg c$ is true or $\neg h$ is true(or both). Thus the truth value of $\neg c$ cannot be determined, the argument is invalid.

□

(d) I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.

∴ I will not buy a new car.

Solution. **Valid**
Assume that

- c: I will buy a new car.
- h: I will buy a new house.
- j: I will get a job.

The expression can be written as below:

$$\begin{array}{l} (c \wedge h) \to j \\ \neg j \\ \underline{h} \\ \therefore \neg c \end{array}$$

1	$(c \wedge h) \to j$	Hypothesis
2	$\neg j$	Hypothesis
3	$\neg(c \wedge h)$	Modus tollens, 1, 2
4	$\neg c \vee \neg h$	De Morgan's law, 3
5	$\neg h \vee \neg c$	Commutative law, 4
6	h	Hypothesis
7	$\neg(\neg h)$	Double negation law, 6
8	$\neg c$	Disjunctive syllogism, 5, 7

Thus the argument is valid.

□

b) Solve the following questions from the Discrete Math zyBook:

1.13.3 Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}.

(b) $\exists x(P(x) \vee Q(x))$
$\exists x \neg Q(x)$

$\therefore \exists x P(x)$

Solution. **Invalid**

	P	Q
a	F	T
b	F	F

For Hypothesis, when $x = a$, $\exists x(P(x) \vee Q(x))$ is true, when $x = b$, $\exists x \neg Q(x)$ is true.
For conclusion, $P(x)$ is false no matter $x = a$ or $x = b$, which means $\exists x P(x)$ is false.
Thus the argument is invalid. □

1.13.5 Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid.

The domain for each problem is the set of students in a class.

(d) Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.
____
Penelope did not get a detention.

Solution. **Invalid**
Assume that,

- $M(x)$: x missed class.
- $S(x)$: x is a student in the class.
- $D(x)$: x got detention.

The expression can be written as below:
$\forall x(M(x) \vee S(x) \to D(x))$
$\exists x S(x)$
$\exists x \neg M(x)$
____
$\therefore \exists x \neg D(x)$
When $x = Penelope$,using **existential instantiation** and **universal instantiation**, the expression becomes:
$M(Penelope) \vee S(Penelope) \to D(Penelope)$
$S(Penelope)$
$\neg M(Penelope)$
____
$\therefore \neg D(Penelope)$

1	$S(Penelope)$	Hypothesis
2	$M(Penelope) \vee S(Penelope)$	Domination law, 1
3	$M(Penelope) \vee S(Penelope) \to D(Penelope)$	Hypothesis
4	$D(Penelope)$	Modus ponens, 2, 3

Thus, the conclusion of the argument contracts with the table above, the argument is invalid.

□

(e) Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.
―――――――――――――――
Penelope did not get a detention.

Solution. **Valid**
Assum that,

- $S(x)$: x is a student in the class.
- $M(x)$: x missed class.
- $D(x)$: x got a detention.
- $G(x)$: x got an A.

The expression can be written as below:
$\forall x(S(x) \wedge (M(x) \vee D(x)) \to \neg G(x))$
$\forall x S(x)$
$\exists x G(x)$
―――――――――――――――
$\therefore \exists x \neg D(x)$
When $x = Penelope$, using **existential instantiation** and **universal instantiation**, the expression becomes:
$S(Penelope) \wedge (M(Penelope) \vee D(Penelope)) \to \neg G(Penelope)$
$S(Penelope)$
$G(Penelope)$
―――――――――――――――
$\therefore \neg D(Penelope)$

1	$S(Penelope) \wedge (M(Penelope) \vee D(Penelope)) \to \neg G(Penelope)$	Hypothesis
2	$G(Penelope)$	Hypothesis
3	$\neg\neg G(Penelope)$	Double negation law, 1
4	$\neg(S(Penelope) \wedge (M(Penelope) \vee D(Penelope)))$	Modus tollens, 1, 3
5	$\neg((S(Penelope) \wedge M(Penelope)) \vee (S(Penelope) \wedge D(Penelope)))$	Distributive law, 4
6	$S(Penelope)$	Hypothesis
7	$\neg(M(Penelope) \vee D(Penelope))$	Identity law, 5, 6
8	$\neg M(Penelope) \wedge \neg D(Penelope)$	De Morgan's law, 7
9	$\neg D(Penelope)$	Simplification, 8

Thus the argument is valid.

$\square$

Question 6

2.4.1 Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k + 1$, where k is an integer. An even integer is an integer that can be expressed as $2k$, where k is an integer.
Prove each of the following statements using a direct proof.

(d) The product of two odd integers is an odd integer.

Direct Proof. Let a be an odd integer, $a = 2k + 1$ where k is an integer. Let b also be an odd integer, $b = 2j + 1$ where j is an integer. Then

$$\begin{aligned} ab &= (2k + 1) * (2j + 1) \\ &= 4kj + 2k + 2j + 1 \\ &= 2(2kj + k + j) + 1 \end{aligned}$$

Since k and j are integers, kj is an integer, $2kj + k + j$ is also an integer. Thus, $ab = 2m + 1$ where m is an integer and $m = 2kj + k + j$, the product of two odd integers is an odd integer.

□

2.4.3 Prove each of the following statements using a direct proof.

(b) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Direct Proof. The binomial can be expressed as below:

$$12 - 7x + x^2 \geq 0$$
$$(x - 3)(x - 4) \geq 0$$

When $x \leq 3$, $x - 3 \leq 0$;
$x - 4 = (x - 3) - 1 \leq 0 - 1$, so $x - 4 < 0$;
Thus $(x - 3)(x - 4) \geq 0$, which means $12 - 7x + x^2 \geq 0$. □

Question 7

2.5.1 Prove each statement by contrapositive.

(d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof by contrapositive. To prove the statement is valid, we can try to prove that if n is even, then $n^2 - 2n + 7$ is odd.

As n is even, let $n = 2k$, where k is an integer.

$$\begin{aligned} n^2 - 2n + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 2(2k^2 - 2k + 3) + 1 \end{aligned}$$

As k is an integer, $2k^2 - 2k + 3$ is also an integer. $n^2 - 2n + 7 = 2m + 1$, where m is an integer and $m = 2k^2 - 2k + 3$, thus $n^2 - 2n + 7$ is odd.

□

2.5.4 Prove each statement by contrapositive

(a) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

Proof by contrapositive. To prove the statement is valid, we can try to prove that if $x > y$, then $x^3 + xy^2 > x^2y + y^3$.

As $x > y$, $x - y > 0$;
As $x > y$, then $x^2 \geq 0$ and $y^2 > 0$, so $x^2 + y^2 > 0$.

$$\begin{aligned} &(x^3 + xy^2) - (x^2y + y^3) \\ &\qquad = x(x^2 + y^2) - y(x^2 + y^2) \\ &\qquad = (x - y)(x^2 + y^2) > 0 \end{aligned}$$

Thus $(x^3 + xy^2) - (x^2y + y^3) > 0$, which means that $x^3 + xy^2 > x^2y + y^3$.

□

(b) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Proof by contrapositive. To prove the statement, we can try to prove that if $x \leq 10$ and $y \leq 10$, then $x + y \leq 20$.

As $x \leq 10$ and $y \leq 10$,

$$\begin{aligned} x &\leq 10 \\ x + y &\leq 10 + y \end{aligned}$$

$$\begin{aligned} y &\leq 10 \\ 10 + y &\leq 10 + 10 \\ 10 + y &\leq 20 \end{aligned}$$

Thus $x + y \leq 10 + y \leq 20$, which means that $x + y \leq 20$.

□

2.5.5 Prove each statement using a direct proof or proof by contrapositive. One method may be much easier than the other.

(c) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Proof by contrapositive. To prove the argument, we can try to prove that if $\frac{1}{x}$ is rational,then x is also rational.

As $\frac{1}{x}$ is rational, note that $\frac{1}{x} \neq 0$, there exists non-zero m and n where m is rational and n is rational, such that $\frac{1}{x} = \frac{m}{n}$.
Thus $x = \frac{n}{m}$ is also rational as both m and n are rational.

□

Question 8

2.6.6 Give a proof for each statement.

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

Direct Proof. Assume that a, b and c are three real numbers and $a \leq b \leq c$, then

$$a + b + c \geq a + a + a$$
$$\frac{a + b + c}{3} \geq \frac{a + a + a}{3}$$
$$\frac{a + b + c}{3} \geq a$$

Thus the average of a, b and c is greater than or equal to a, which means the argument is valid.

□

(d) There is no smallest integer.

Proof by contradiction. Assume that there exists the smallest integer s, then $s - 1$ is also integer, and $s - 1 < s$, which contradicts the premise that s is the smallest integer. Thus the argument is valid. □

Question 9

2.7.2 Prove each statement. If integers x and y have the same parity, then $x + y$ is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Proof. As x and y have the same parity, x and y either both are odd or both are even.

Case 1: x and y are both odd.
Since x and y are both odd, there exists an integer j such that $x = 2j+1$, there exists an integer k such that $y = 2k + 1$.

$$\begin{aligned} x + y &= 2j + 1 + 2k + 1 \\ &= 2(j + k + 1) \end{aligned}$$

Since j and k are integers, $j + k + 1$ is also an integer. Therefore $x + y$ is even.

Case 2. x and y are both even.
Since x and y are both even, there exists an integer j such that $x = 2j$, there exists an integer k such that $y = 2k$.

$$\begin{aligned} x + y &= 2j + 2k \\ &= 2(j + k) \end{aligned}$$

Since j and k are integers, $j + k$ is also an integer. Therefore $x + y$ is even.

□