Homework 2

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Question 7

Solve the following questions from the Discrete Math zyBook:

3.1.1 Use the definitions for the sets given below to determine whether each statement is true or false:

$A = \{x \in \mathbb{Z} : x \text{ is an integer multiple of } 3\}$
$B = \{x \in \mathbb{Z} : x \text{ is a perfect square}\}$
$C = \{4, 5, 9, 10\}$
$D = \{2, 4, 11, 14\}$
$E = \{3, 6, 9\}$
$F = \{4, 6, 16\}$

An integer x is a perfect square if there is an integer y such that $x = y^2$.

(a) $27 \in A$

Solution. **True.**
Since $27 = 3 * 9$, 27 is an element of set A, $27 \in A$. □

(b) $27 \in B$

Solution. **False.**
Since $\sqrt{27} = 3\sqrt{3}$, and $3\sqrt{3}$ is not an integer, by definition of perfect square, $27 \notin B$. □

(c) $100 \in B$

Solution. **True.**
Since $100 = 10^2$ and 10 is an integer, by definition of perfect square, $100 \in B$. □

(d) $E \subseteq C$ or $C \subseteq E$

Solution. **False.**
4, 5 and 10 are elements in C but not in E.

3 and 6 are elements not in C but in E.

It implies that neither C is a subset of E nor E is a subset of C. □

(e) $E \subseteq A$

Solution. **True.**
For each element in E, $3 = 1 * 3$, $6 = 2 * 3$, $9 = 3 * 3$. By the definition of A, E is a subset of A. □

(f) $A \subset E$

Solution. **False.**
By the definition of A and $30 = 10 * 3$, 30 is one of the elements of A, but 30 is not in E, which implies that E does not contain all the elements of A, $A \not\subset E$. □

(g) $E \in A$

Solution. **False.**
By the definition of A, all elements are integers, while E is a set, thus E is not an element of A. □

3.1.2 Use the definitions for the sets given below to determine whether each statement is true or false:

$A = \{x \in \mathbb{Z} : x \text{ is an integer multiple of } 3\ \}$

$B = \{x \in \mathbb{Z} : x \text{ is a perfect square}\}$

$C = \{4, 5, 9, 10\}$

$D = \{2, 4, 11, 14\}$

$E = \{3, 6, 9\}$

$F = \{4, 6, 16\}$

An integer x is a perfect square if there is an integer y such that $x = y^2$.

(a) $15 \subset A$

Solution. **False.**
Since the integer $15 = 5 * 3$, 15 is an element of A, but not a subset of A. □

(b) $\{15\} \subset A$

Solution. **True.**
Since 15 is the only element in $\{15\}$ and $15 = 5*3$, by the definition of A, $\{15\}$ is a subset of A. □

(c) $\varnothing \subset C$

Solution. **True.**
$\varnothing$ is a subset of any set, thus it is a subset of C. □

(d) $D \subseteq D$

Solution. **True.**
Any set is a subset of itself, thus $D \subseteq D$. □

(e) $\varnothing \in B$

Solution. **False.**
By the definition of B, All elements of B are integers, and $\varnothing$ is an empty set, not an integer, thus $\varnothing$ is not an element of B. □

3.1.5 Express each set using set builder notation. Then if the set is finite, give its cardinality. Otherwise, indicate that the set is infinite.

(b) $\{3, 6, 9, 12, ...\}$

Solution. **Infinite.**
$\{x \in \mathbb{Z}^+ : x \text{ is an integer multiple of } 3\}$
Every element in the set is a positive integer multiple of 3, as integer is infinite, the number of elements in the set is also infinite. □

(d) $\{0, 10, 20, 30, ..., 1000\}$

Solution. $|\{\mathbf{0}, \mathbf{10}, \mathbf{20}, \mathbf{30}, ..., \mathbf{1000}\}| = \mathbf{101}$.
$\{x \in \mathbb{N} : x \text{ is an integer multiple of } 10 \text{ and } x \leq 1000\}$
Every element in the set is an integer multiple of 10 in the range of $[0, 1000]$, thus the cardinality of the set is $1 + 1000/10 = 101$. □

3.2.1 Let $X = \{1, \{1\}, \{1, 2\}, 2, \{3\}, 4\}$. Which statements are true?

(a) $2 \in X$

Solution. **True.** Clearly 2 is an integer and is an element of X. □

(b) $\{2\} \subseteq X$

Solution. **True.** $\{2\}$ is a set with 2 as the element, and 2 is also an element of X. □

(c) $\{2\} \in X$

Solution. **False.** By definition, $\{2\}$ is not an element of X. □

(d) $3 \in X$

Solution. **False.** By definition, 3 is not an element of X. □

(e) $\{1, 2\} \in X$

Solution. **True.** By definition, $\{1, 2\}$ is an element of X. □

(f) $\{1, 2\} \subseteq X$

Solution. **True.** By definition, 1 is an element of X and so is 2, thus the set $\{1, 2\}$ is a subset of X. □

(g) $\{2, 4\} \subseteq X$

Solution. **True.** By definition, 2 is an element of X and so is 4, thus the set $\{2, 4\}$ is a subset of X. □

(h) $\{2, 4\} \in X$

Solution. **False.** By definition, $\{2, 4\}$ is not an element of X. □

(i) $\{2, 3\} \subseteq X$

Solution. **False.** By definition, 2 is an element of X but 3 is not an element of X, thus $\{2, 3\}$ is not a subset of X. □

(j) $\{2, 3\} \in X$

Solution. **False.** By definition, $\{2,\ 3\}$ is not an element of X. □

(k) $|X| = 7$

Solution. **False.** Clearly there are 6 elements in X, not 7. □

Question 8

3.2.4 Let $A = \{1, 2, 3\}$. What is $\{X \in P(A) : 2 \in X\}$?

Solution. Namely X is an element of the power set of A such that 2 is an element of X. Since $A = \{1, 2, 3\}$, therefore we have

$$X = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$$

□

Question 9

Solve the following questions from the Discrete Math zyBook:

3.3.1 Define the sets A, B, C, and D as follows:

$$A = \{-3, 0, 1, 4, 17\}$$
$$B = \{-12, -5, 1, 4, 6\}$$
$$C = \{x \in \mathbb{Z} : x \text{ is odd}\}$$
$$D = \{x \in \mathbb{Z} : x \text{ is positive}\}$$

For each of the following set expressions, if the corresponding set is finite, express the set using roster notation. Otherwise, indicate that the set is infinite.

(c) $A \cap C$

Solution.

$$\{-3, 1, 17\}$$

□

(d) $A \cup (B \cap C)$

Solution.

$$\{-5, -3, 0, 1, 4, 17\}$$

□

(e) $A \cap B \cap C$

Solution.

$$\{1\}$$

□

3.3.3 Use the following definitions to express each union or intersection given. You can use roster or set builder notation in your responses, but no set operations. For each definition, $i \in \mathbb{Z}^+$.

- $A_i = \{i^0, i^1, i^2\}$(Recall that for any number $x, x^0 = 1$.)
- $B_i = \{x \in \mathbb{R} : -i \leq x \leq 1/i\}$
- $C_i = \{x \in \mathbb{R} : -1/i \leq x \leq 1/i\}$

(a) $\bigcap_{i=2}^{5} A_i$

Solution.

$$\begin{aligned}\bigcap_{i=2}^{5} A_i &= \{2^0, 2^1, 2^2\} \cap \{3^0, 3^1, 3^2\} \cap \{4^0, 4^1, 4^2\} \cap \{5^0, 5^1, 5^2\} \\ &= \{1, 2, 4\} \cap \{1, 3, 9\} \cap \{1, 4, 16\} \cap \{1, 5, 25\} \\ &= \{1\}\end{aligned}$$

□

(b) $\bigcup_{i=2}^{5} A_i$

Solution.

$$\begin{aligned}\bigcup_{i=2}^{5} A_i &= \{2^0, 2^1, 2^2\} \cup \{3^0, 3^1, 3^2\} \cup \{4^0, 4^1, 4^2\} \cup \{5^0, 5^1, 5^2\} \\ &= \{1, 2, 4\} \cup \{1, 3, 9\} \cup \{1, 4, 16\} \cup \{1, 5, 25\} \\ &= \{1, 2, 3, 4, 5, 9, 16, 25\}\end{aligned}$$

□

(e) $\bigcap_{i=1}^{100} C_i$

Solution. By the definition of C_i, as i increases from 0 to 100, the range of C_i narrows, and $C_i \subseteq C_j$ for $i \geq j$

$$C_1 = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$$

$$\begin{aligned}C_2 &= \{x \in \mathbb{R} : -\frac{1}{2} \leq x \leq \frac{1}{2}\} \\ C_3 &= \{x \in \mathbb{R} : -\frac{1}{3} \leq x \leq \frac{1}{3}\} \\ &\dots \\ C_{100} &= \{x \in \mathbb{R} : -\frac{1}{100} \leq x \leq \frac{1}{100}\}\end{aligned}$$

Therefore,

$$\begin{aligned}\bigcap_{i=1}^{100} C_i &= C_{100} \\ &= \{x \in \mathbb{R} : \frac{-1}{100} \leq x \leq \frac{1}{100}\}\end{aligned}$$

□

(f) $\bigcup_{i=1}^{100} C_i$

Solution. By the definition of C_i, as i increases from 0 to 100, the range of C_i narrows, and $C_i \subseteq C_j$ for $i \geq j$

$$C_1 = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$$

$$\begin{aligned} C_2 &= \{x \in \mathbb{R} : -\frac{1}{2} \leq x \leq \frac{1}{2}\} \\ C_3 &= \{x \in \mathbb{R} : -\frac{1}{3} \leq x \leq \frac{1}{3}\} \\ &... \\ C_{100} &= \{x \in \mathbb{R} : -\frac{1}{100} \leq x \leq \frac{1}{100}\} \end{aligned}$$

Therefore,

$$\begin{aligned} \bigcup_{i=1}^{100} C_i &= C_1 \\ &= \{x \in \mathbb{R} : -1 \leq x \leq 1\} \end{aligned}$$

□

3.3.4 Use the set definitions $A = \{a, b\}$ and $B = \{b, c\}$ to express each set below. Use roster notation in your solutions.

(b) $P(A \cup B)$

Solution.

$$\begin{aligned} P(A \cup B) &= P(\{a, b\} \cup \{b, c\}) \\ &= P(\{a, b, c\}) \\ &= \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \end{aligned}$$

□

(d) $P(A) \cup P(B)$

Solution.

$$\begin{aligned} P(A) \cup P(B) &= P(\{a, b\}) \cup P(\{b, c\}) \\ &= \{\varnothing, \{a\}, \{b\}, \{a, b\}\} \cup \{\varnothing, \{b\}, \{c\}, \{b, c\}\} \\ &= \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\} \end{aligned}$$

□

Question 10

Solve the following questions from the Discrete Math zyBook:

3.5.1 The sets A, B, and C are defined as follows:

$$A = \{\text{tall, grande, venti}\}$$
$$B = \{\text{foam, no-foam}\}$$
$$C = \{\text{non-fat, whole}\}$$

Use the definitions for A, B, and C to answer the questions. Express the elements using n-tuple notation, not string notation.

(b) Write an element from the set $B \times A \times C$.

Solution. One of the elements is (foam, tall, whole) □

(c) Write the set $B \times C$ using roster notation.

Solution. $B \times C = \{$(foam, non-fat), (no-foam, non-fat), (foam, whole), (no-foam, whole)$\}$ □

3.5.3 Indicate which of the following statements are true.

(b) $\mathbb{Z}^2 \subseteq \mathbb{R}^2$

Solution. **True.**
By definition, $\mathbb{Z} \subseteq \mathbb{R}$, if $(x, y) \in \mathbb{Z}^2$, then $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$, which implies that $x \in \mathbb{R}$ and $y \in \mathbb{R}$, thus $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ is true. □

(c) $\mathbb{Z}^2 \cap \mathbb{Z}^3 = \varnothing$

Solution. **True.**
Every element of $\mathbb{Z}^2$ is a pair of numbers, while every element of $\mathbb{Z}^3$ is a triple, thus there are no common elements in these two sets, $\mathbb{Z}^2 \cap \mathbb{Z}^3 = \varnothing$. □

(e) For any three sets, A, B, and C, if $A \subseteq B$, then $A \times C \subseteq B \times C$

Solution. **True.**
If $(x, y) \in A \times C$, then $x \in A$ and $y \in C$. Since $A \subseteq B$, then $x \in B$, thus $(x, y) \in B \times C$. □

3.5.6 Express the following sets using the roster method. Express the elements as strings, not n-tuples.

(d) $\{xy$: where $x \in \{0\} \cup \{0\}^2$ and $y \in \{1\} \cup \{1\}^2$ $\}$

Solution. By definition, $x \in \{0, 00\}$ and $y \in \{1, 11\}$. Thus xy is an element of the set

$$\{01, 011, 001, 0011\}$$

□

(e) $\{xy$: $x \in \{aa,\ ab\}$ and $y \in \{a\} \cup \{a\}^2\}$

Solution. By definition, $y \in \{a, aa\}$, since $x \in \{aa, ab\}$, thus xy is an element of the set

{aaa, aaaa, aba, abaa}

□

3.5.7 Use the following set definitions to specify each set in roster notation. Except where noted, express elements of Cartesian products as strings.

- $A = \{a\}$
- $B = \{b, c\}$
- $C = \{a, b, d\}$

(c) $(A \times B) \cup (A \times C)$

Solution.

$$\begin{aligned}(A \times B) \cup (A \times C) &= \{ab, ac\} \cup \{aa, ab, ad\} \\ &= \{aa, ab, ac, ad\}\end{aligned}$$

□

(f) $P(A \times B)$

Solution.

$$\begin{aligned}P(A \times B) &= P(\{ab, ac\}) \\ &= \{\varnothing, \{ab\}, \{ac\}, \{ab, ac\}\}\end{aligned}$$

□

(g) $P(A) \times P(B)$. Use ordered pair notation for elements of the Cartesian product.

Solution.

$$
\begin{aligned}
&(A) \times P(B) \\
&= \{\varnothing, \{a\}\} \times \{\varnothing, \{b\}, \{c\}, \{b, c\}\} \\
&= \{(\varnothing, \varnothing), (\varnothing, \{b\}), (\varnothing, \{c\}), (\varnothing, \{b, c\}), (\{a\}, \varnothing), (\{a\}, \{b\}), (\{a\}, \{c\}), (\{a\}, \{b, c\})\}
\end{aligned}
$$

□

Question 11

Solve the following questions from the Discrete Math zyBook:

3.6.2 Use the set identities given in the table to prove the following new identities. Label each step in your proof with the set identity used to establish that step.

(b) $(B \cup A) \cap (\overline{B} \cup A) = A$

Solution.

$(B \cup A) \cap (\overline{B} \cup A)$	
$(B \cap \overline{B}) \cup A$	Distributive law
$\varnothing \cup A$	Complement law
A	Identity law

□

(c) $\overline{A \cap \overline{B}} = \overline{A} \cup B$

Solution.

$\overline{A \cap \overline{B}}$	
$\overline{A} \cup \overline{\overline{B}}$	De Morgan's law
$\overline{A} \cup B$	Double complement law

□

3.6.3 A set equation is not an identity if there are examples for the variables denoting the sets that cause the equation to be false. For example $A \cup B = A \cap B$ is not an identity because if $A = \{1, 2\}$ and $B = \{1\}$, then $A \cup B = \{1, 2\}$ and $A \cap B = \{1\}$, which means that $A \cup B \neq A \cap B$. Show that each set equation given below is not a set identity.

(b) $A - (B \cap A) = A$

Solution. If $A = \{1, 2\}$ and $B = \{2, 3\}$, then $B \cap A = \{2\}$, $A - (B \cap A) = \{1\}$, which means that $A - (B \cap A) \neq A$. □

(d) $(B - A) \cup A = A$

Solution. If $A = \{1, 2\}$ and $B = \{2, 3\}$, then $B - A = \{3\}$, $(B - A) \cup A = \{1, 2, 3\}$, which means that $(B - A) \cup A \neq A$ □

3.6.4 The set subtraction law states that $A - B = A \cap \overline{B}$. Use the set subtraction law as well as the other set identities given in the table to prove each of the following new identities. Label each step in your proof with the set identity used to establish that step.

(b) $A \cap (B - A) = \varnothing$

Solution.

$A \cap (B - A)$	
$A \cap (B \cap \overline{A})$	Set subtraction law
$A \cap (\overline{A} \cap B)$	Commutative law
$(A \cap \overline{A}) \cap B$	Associative law
$\varnothing \cap B$	Complement law
$\varnothing$	Domination law

□

(c) $A \cup (B - A) = A \cup B$

Solution.

$A \cup (B - A)$	
$A \cup (B \cap \overline{A})$	Set subtraction law
$(A \cup B) \cap (A \cup \overline{A})$	Distributive law
$(A \cup B) \cap U$	Complement law
$A \cup B$	Identity law

□