Homework 4

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Question 9

Solve the following questions from the Discrete Math zyBook:

4.1.3 Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

(b)

$$f(x) = \frac{1}{(x^2 - 4)}$$

Solution. The function is not well-defined for $x = 2$ and $x = -2$. □

(c)

$$f(x) = \sqrt{x^2}$$

Solution. The function is well-defined. And the range of the function is $\mathbb{R}^+ \cup \{0\}$.

□

4.1.5 Express the range of each function using roster notation.

(b) Let $A = \{2, 3, 4, 5\}$.
$f : A \to \mathbb{Z}$ such that $f(x) = x^2$

Solution.

$$\{4, 9, 16, 25\}$$

□

(d) $f : \{0, 1\}^5 \to \mathbb{Z}$. For $x \in \{0, 1\}^5$, $f(x)$ is the number of 1's that occur in x. For example, $f(01101) = 3$, because there are three 1's in the string "01101".

Solution.

$$\{0, 1, 2, 3, 4, 5\}$$

□

(h) Let $A = \{1, 2, 3\}$
$f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$.

Solution.

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

□

(i) Let $A = \{1, 2, 3\}$
$f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$.

Solution.

$$\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$$

□

(l) Let $A = \{1, 2, 3\}$
$f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - \{1\}$.

Solution.

$$\{\varnothing, \{2\}, \{3\}, \{2, 3\}\}$$

□

Question 10

I Solve the following questions from the Discrete Math zyBook:

4.2.2 For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(c) $h : \mathbb{Z} \to \mathbb{Z}$. $h(x) = x^3$

Solution. The function is **one-to-one, but not onto**.
For example, there is no integer x such that $h(x) = 5$.

□

(g) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 1, 2y)$

Solution. The function is **one-to-one, but not onto**.
For example, there is no integer pair (x, y) such that $f(x, y) = (2, 3)$.

□

(k) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x, y) = 2^x + y$

Solution. The function is **neither one-to-one nor onto**.
For example, $f(0, 7) = 8$ and $f(3, 0) = 8$, $f(0, 7) = f(3, 0)$, and there is no positive integer pair (x, y) such that $f(x, y) = 1$.

□

4.2.4 For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(b) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.

Solution. The function is **neither one-to-one nor onto**.
For example, $f(001) = 101$ and $f(101) = 101$, $f(001) = f(101)$, and there is no input string $s \in \{0, 1\}^3$ such that $f(s) = 001$.

□

(c) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.

Solution. The function is **both one-to-one and onto**.

□

(d) $f : \{0, 1\}^3 \to \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $f(100) = 1001$.

Solution. The function is **one-to-one, but not onto**.
For example, there is no input string $s \in \{0,1\}^3$ such that $f(s) = 1000$.

□

(g) Let A be defined to be the set $\{1,2,3,4,5,6,7,8\}$ and let $B = \{1\}$. $f{:}P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - B$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.

Solution. The function is **neither one-to-one nor onto**.
For example, $f(\{1,2,3\}) = \{2,3\}$ and $f(\{2,3\}) = \{2,3\}$, $f(\{1,2,3\}) = f(\{2,3\})$, and there is no set $S \in P(A)$ such that $f(S) = \{1\}$.

□

II Give an example of a function from the set of integers to the set of positive integers that is :

a. one-to-one, but not onto.

Solution.

$$f(x) = \begin{cases} 2x+7, & \text{if } x \geq 0, \\ -2x, & \text{if } x < 0. \end{cases}$$

□

b. onto, but not one-to-one.

Solution.

$$f(x) = \begin{cases} x+1, & \text{if } x > 0, \\ -x+1, & \text{if } x \leq 0 \end{cases}$$

□

c. one-to-one and onto.

Solution.

$$f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

□

d. neither one-to-one nor onto

Solution.

$$f(x) = 5$$

□

Question 11

Solve the following questions from the Discrete Math zyBook:

4.3.2 For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

(c) $f : \mathbb{R} \to \mathbb{R}. f(x) = 2x + 3$

Solution. The function is both one-to-one and onto, thus it has a well-defined inverse.

$$f^{-1}(x) = \frac{x - 3}{2}$$

□

(d) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$
$f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
For $X \subseteq A, f(X) = |X|$. Recall that for a finite set $A, P(A)$ denotes the power set of A which is the set of all subsets of A.

Solution. The function is not one-to-one as $f(\{1\}) = f(\{2\})$, thus it doesn't have a well-defined inverse.

□

(g) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$.

Solution. The function has a well-defined inverse $f^{-1} = f$. That is, the output of f^{-1} is obtained by taking the input string and reversing the bits.

□

(i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$

Solution. The function has a well-defined inverse

$$f^{-1}(x, y) = (x - 5, y + 2)$$

□

4.4.8 The domain and target set of functions f, g, and h are $\mathbf{Z}$. The functions are defined as:

- $f(x) = 2x + 3$
- $g(x) = 5x + 7$
- $h(x) = x^2 + 1$

Give an explicit formula for each function given below.

(c) $f \circ h$

Solution.

$$f \circ h(x) = 2x^2 + 5$$

□

(d) $h \circ f$

Solution.

$$h \circ f(x) = 4x^2 + 12x + 10$$

□

4.4.2 Consider three functions f, g, and h, whose domain and target are $\mathbf{Z}$. Let

$$f(x) = x^2 \quad g(x) = 2^x \quad h(x) = \left\lceil \frac{x}{5} \right\rceil$$

(b) Evaluate $(f \circ h)(52)$

Solution.

$$(f \circ h)(52) = (\left\lceil \frac{52}{5} \right\rceil)^2 = 11^2 = 121$$

□

(c) Evaluate $(g \circ h \circ f)(4)$

Solution.

$$(g \circ h \circ f)(4) = (g \circ h)(16) = g(4) = 16$$

□

(d) Give a mathematical expression for $h \circ f$.

Solution.

$$h \circ f(x) = \left\lceil \frac{x^2}{5} \right\rceil$$

□

4.4.6 Define the following functions f, g, and h:

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
- $g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
- $h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

(c) What is $(h \circ f)(010)$?

Solution.

$$(h \circ f)(010) = h(110) = 111$$

□

(d) What is the range of $h \circ f$?

Solution. The range of f is {100, 101, 110, 111}. Thus the range of $h \circ f$ is {101, 111} □

(e) What is the range of $g \circ f$?

Solution. The range of f is {100, 101, 110, 111}. Thus the range of $g \circ f$ is {001, 101, 011, 111} □

Extra Question

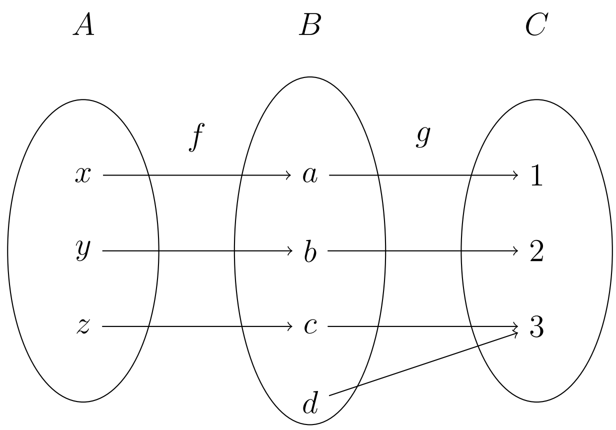
4.4.4 Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Solution. **No.**
Since there exist x_1 and x_2 such that $f(x_1) = f(x_2)$, we have $(g \circ f)(x_1) = (g \circ f)(x_2)$, which means $g \circ f$ is not one-to-one. □

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Solution. **Yes.**
The diagram below illustrates an example.



□