

Homework 6

Wei Ye

2023-08-13

Question 7

Solve the following questions from the Discrete Math zyBook:

6.1.5 A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

- (b) What is the probability that the hand is a three of a kind? A three of a kind has 3 cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank.

Solution.

$$\frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}} = \frac{88}{4165} \approx 0.0211$$

□

- (c) What is the probability that all 5 cards have the same suit?

Solution.

$$\frac{\binom{4}{1} \binom{13}{5}}{\binom{52}{5}} = \frac{33}{16660} \approx 0.0020$$

□

- (d) What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair.

Solution.

$$\frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}} = \frac{352}{833} \approx 0.4226$$

□

6.2.4 A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(a) The hand has at least one club.

Solution.

$$1 - \frac{\binom{39}{5}}{\binom{52}{5}} = \frac{7411}{9520} \approx 0.7785$$

□

(b) The hand has at least two cards with the same rank.

Solution.

$$1 - \frac{\binom{13}{5} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}} = \frac{2053}{4165} \approx 0.4929$$

□

(c) The hand has exactly one club or exactly one spade.

Solution.

$$\frac{2 \cdot \binom{13}{1} \binom{39}{4} - \binom{13}{1} \binom{13}{1} \binom{26}{3}}{\binom{52}{5}} = \frac{65351}{99960} \approx 0.6538$$

□

(d) The hand has at least one club or at least one spade.

Solution.

$$1 - \frac{\binom{26}{5}}{\binom{52}{5}} = \frac{9743}{9996} \approx 0.9747$$

□

Question 8

Solve the following questions from the Discrete Math zyBook:

6.3.2 The letters $\{a, b, c, d, e, f, g\}$ are put in a random order. Each permutation is equally likely. Define the following events:

- A: The letter b falls in the middle (with three before it and three after it)
- B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "agbdcef" would be an outcome in this event.
- C: The letters "def" occur together in that order (e.g. "gdefbca")

(a) Calculate the probability of each individual event. That is, calculate $p(A)$, $p(B)$, and $p(C)$.

Solution.

$$p(A) = \frac{6!}{7!} = \frac{1}{7}$$

$$p(B) = \frac{5! \cdot (6 + 5 + 4 + 3 + 2 + 1)}{7!} = \frac{1}{2}$$

$$p(C) = \frac{5!}{7!} = \frac{1}{42}$$

□

(b) What is $p(A|C)$?

Solution.

$$p(A|C) = \frac{|A \cap C|}{|C|} = \frac{2! \cdot 3!}{5!} = \frac{1}{10}$$

□

(c) What is $p(B|C)$?

Solution.

$$p(B|C) = \frac{|B \cap C|}{|C|} = \frac{3! \cdot (4 + 3 + 2 + 1)}{5!} = \frac{1}{2}$$

□

(d) What is $p(A|B)$?

Solution.

$$p(A|B) = \frac{|A \cap B|}{|B|} = \frac{\binom{3}{1} \cdot 5!}{5! \cdot (6 + 5 + 4 + 3 + 2 + 1)} = \frac{1}{7}$$

□

(e) Which pairs of events among A , B , and C are independent?

Solution.

A and B is independent, as $P(A) = P(A|B)$

A and C is not independent, as $P(A) \neq P(A|C)$

B and C is independent, as $P(B) = P(B|C)$

□

6.3.6 A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is $1/3$ and the probability of tails is $2/3$. The outcomes of the coin flips are mutually independent. What is the probability of each event?

(b) The first 5 flips come up heads. The last 5 flips come up tails.

Solution.

$$\left(\frac{1}{3}\right)^5 \cdot \left(\frac{2}{3}\right)^5 = \frac{32}{59049} \approx 0.0005$$

□

(c) The first flip comes up heads. The rest of the flips come up tails.

Solution.

$$\left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^9 = \frac{512}{59049} \approx 0.0087$$

□

6.4.2 (a) Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1, 2, 3, 4 and 5. You choose a die at random, and roll it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

Solution. Let F be the event that the fair dice is chose.

Let Y be the event that the outcome is 4,3,6,6,5,5 when rolling a dice six times.

Then we shall calculate the probability of $p(F|Y)$.

$$p(F) = p(\bar{F}) = \frac{1}{2}$$

$$p(Y|F) = \left(\frac{1}{6}\right)^6$$

$$p(Y|\bar{F}) = (0.15)^4 \cdot (0.25)^2$$

Therefore, according to Bayes' theorem, we have

$$\begin{aligned} p(F|Y) &= \frac{p(Y|F) \cdot p(F)}{p(Y|F) \cdot p(F) + p(Y|\bar{F}) \cdot p(\bar{F})} \\ &= \frac{\left(\frac{1}{6}\right)^6 \cdot \frac{1}{2}}{\left(\frac{1}{6}\right)^6 \cdot \frac{1}{2} + (0.15)^4 \cdot (0.25)^2 \cdot \frac{1}{2}} \\ &\approx 0.4038 \end{aligned}$$

□

Question 9

Solve the following questions from the Discrete Math zyBook:

6.5.2 A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

(a) What is the range of A ?

Solution. $\{0, 1, 2, 3, 4\}$ □

(b) Give the distribution over the random variable A .

Solution.

$$\text{When } A = 0, p(A) = \frac{\binom{48}{5}}{\binom{52}{5}} \approx 0.6588$$

$$\text{When } A = 1, p(A) = \frac{\binom{4}{1} \binom{48}{4}}{\binom{52}{5}} \approx 0.2995$$

$$\text{When } A = 2, p(A) = \frac{\binom{4}{2} \binom{48}{3}}{\binom{52}{5}} \approx 0.0399$$

$$\text{When } A = 3, p(A) = \frac{\binom{4}{3} \binom{48}{2}}{\binom{52}{5}} \approx 0.0017$$

$$\text{When } A = 4, p(A) = \frac{\binom{4}{4} \binom{48}{1}}{\binom{52}{5}} \approx 0.000018$$

Therefore, the distribution of A is

$$(0, 0.6588), (1, 0.2995), (2, 0.0399), (3, 0.0017), (4, 0.000018)$$

□

6.6.1 (a) Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is $E[G]$?

Solution. The range of G is $\{0, 1, 2\}$

The distribution of G is

$$\left(0, \frac{\binom{3}{2}}{\binom{10}{2}}\right), \left(1, \frac{\binom{7}{1} \binom{3}{1}}{\binom{10}{2}}\right), \left(2, \frac{\binom{7}{2}}{\binom{10}{2}}\right)$$

Therefore, the expectation of G is

$$E[G] = 0 \cdot \frac{\binom{3}{2}}{\binom{10}{2}} + 1 \cdot \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} + 2 \cdot \frac{\binom{7}{2}}{\binom{10}{2}} = \frac{7}{5}$$

□

- 6.6.4 (a) A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then $X = 25$. What is $E[X]$?

Solution. The range of X is $\{1, 4, 9, 16, 25, 36\}$
The distribution of X is

$$\left(1, \frac{1}{6}\right), \left(4, \frac{1}{6}\right), \left(9, \frac{1}{6}\right), \left(16, \frac{1}{6}\right), \left(25, \frac{1}{6}\right), \left(36, \frac{1}{6}\right)$$

Therefore, the expectation of X is

$$E(X) = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6} = \frac{91}{6}$$

□

- (b) A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and $Y = 4$. What is $E[Y]$?

Solution. The range of Y is $\{0, 1, 4, 9\}$
The distribution of Y is

$$\left(0, \left(\frac{1}{2}\right)^3\right), \left(1, \binom{3}{1}\left(\frac{1}{2}\right)^3\right), \left(4, \binom{3}{2}\left(\frac{1}{2}\right)^3\right), \left(9, \left(\frac{1}{2}\right)^3\right)$$

Therefore, the expectation of Y is

$$E(Y) = 0 \cdot \left(\frac{1}{2}\right)^3 + 1 \cdot \binom{3}{1}\left(\frac{1}{2}\right)^3 + 4 \cdot \binom{3}{2}\left(\frac{1}{2}\right)^3 + 9 \cdot \left(\frac{1}{2}\right)^3 = 3$$

□

- 6.7.4 (a) A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?

Solution. Let X_i be the event that i -th student who got his own coat, for $i \geq 1$ and $i \leq 10$, $p(X_i) = \frac{9!}{10!} = \frac{1}{10}$. Therefore the expected number of children who get his or her own coat is

$$\sum_{i=1}^{10} X_i = 10 \cdot \frac{1}{10} = 1$$

□

Question 10

Solve the following questions from the Discrete Math zyBook:

6.8.1 The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

- (a) What is the probability that out of 100 circuit boards made exactly 2 have defects?

Solution. Let Y be the event that out of 100 circuit boards made exactly 2 have defects.

$$\begin{aligned} p(Y) &= b(2; 100, 0.01) \\ &= \binom{100}{2} \cdot (0.01)^2 \cdot (1 - 0.01)^{100-2} \\ &= \binom{100}{2} \cdot (0.01)^2 \cdot (0.99)^{98} \end{aligned}$$

□

- (b) What is the probability that out of 100 circuit boards made at least 2 have defects?

Solution. Let Y be the event that out of 100 circuit boards made at least 2 have defects.

$$\begin{aligned} p(Y) &= 1 - b(0; 100, 0.01) - b(1; 100, 0.01) \\ &= 1 - (1 - 0.01)^{100} - \binom{100}{1} \cdot (0.01)^1 \cdot (1 - 0.01)^{99} \\ &= 1 - 0.99^{100} - 0.99^{99} \end{aligned}$$

□

- (c) What is the expected number of circuit boards with defects out of the 100 made?

Solution. $100 \cdot 0.01 = 1$

□

- (d) Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compare to the situation in which each circuit board is made separately?

Solution. Let X be the event that out of 100 circuit boards at least 2 have defects.

$$p(X) = 1 - p(0; 50, 0.01) = 1 - (1 - 0.01)^{50} = 1 - 0.99^{50}$$

The expected number of circuit boards with defects out of the 50 pairs made is

$$50 \cdot 0.01 \cdot 2 = 1$$

Thus, the expected number of circuit boards with defects is the same no matter they are made separately or in pairs.

However, as $p(Y) < p(X)$, the probability with defects is higher if we make circuit boards in pairs/batches.

□

6.8.3 A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

- (b) What is the probability that you reach an incorrect conclusion if the coin is biased?

Solution. If the coin is biased, when the number of heads is at least 4, the conclusion is incorrect.

$$1 - 0.7^{10} - \binom{10}{1} \cdot 0.3 \cdot 0.7^9 - \binom{10}{2} \cdot 0.3^2 \cdot 0.7^8 - \binom{10}{3} \cdot 0.3^3 \cdot 0.7^7 \approx 0.3504$$

□