Homework 7

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Question 3

a. 8.2.2 Give complete proofs for the growth rates of the polynomials below. You should provide specific values for c and n_0 and prove algebraically that the functions satisfy the definitions for O and Ω.

   b) $f(n) = n^3 + 3n^2 + 4$. Prove that $f = \Theta(n^3)$.

   Solution. Let $c_1 = 1$, $c_2 = 8$ and $n_0 = 1$, then for any $n \geq n_0$ we have

$$1 \cdot n^3 \leq n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3 = 8n^3$$

$$n^3 \leq n^3 + 3n^2 + 4 \leq 8n^3$$

   Thus $f = \Theta(n^3)$.

□

b. Use the definition of Θ to show that $\sqrt{7n^2 + 2n - 8} = \Theta(n)$

Solution. For the expression inside the square root, let $c_1 = 7$, $c_2 = 9$ and $n_0 = 4$, then for any $n \geq n_0$, we have

$$7n^2 \leq 7n^2 + 2n - 8 \leq 7n^2 + 2n^2$$

$$7n^2 \leq 7n^2 + 2n - 8 \leq 9n^2$$

Thus $7n^2 + 2n - 8 = \Theta(n^2)$
Therefore, $\sqrt{7n^2 + 2n - 8} = \Theta(\sqrt{n^2}) = \Theta(n)$

□

c. 8.3.5 The algorithm below makes some changes to an input sequence of numbers.

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MysteryAlgorithm

Input: a_1, a_2, ..., a_n
       n, the length of the sequence.
       p, a number.
Output: ??

i := 1
j := n

While (i < j)
     While (i < j and a_i < p)
          i := i + 1
     End-while
     While (i < j and a_i ≥ p)
          j := j − 1
     End-while
     if (i < j), swap a_i and a_j
End-while

Return (a_1, a_2, ..., a_n)
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(a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with $p = 0$)

Solution. This is a two-pointer algorithm separating a sequence of numbers into two blocks, one with numbers smaller than p and the other with numbers equal to or larger than p.

There are two pointers i and j pointing to the i-th and j-th number in the sequence.

Pointer i moves from left to right starting from the beginning of the number sequence, and pointer j moves from right to left starting from the end of the number sequence.

Step One – If the i-th number in the sequence is smaller than p, pointer i keeps moving, until the i-th number is not smaller than

p.

Step Two – Then if the j-th number in the sequence is equal to or larger than p, pointer j keeps moving, until the j-th number is smaller than p.

Step Three – If the two pointers that move in opposite directions haven't met each other, swap the i-th number and j-th number, and then repeat step one and step two until the two pointers point to the same number, which means that all the numbers in the sequence have been iterated through.

□

(b) What is the total number of times that the lines "$i := i + 1$" or "$j := j - 1$" are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

Solution. The total number of times that the lines are executed on a sequence of length n is $n - 1$. □

(c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

Solution. The total number of times that the swap operation is executed depends on the actual values of the numbers in the sequence.

When all the numbers that are smaller than p are gathered in the left and numbers that are equal to or larger than p are gathered in the right, no matter each block has how many numbers, the swap operation has the minimum number of times of execution, which equals to 0.

On the contrary, when all the numbers that are smaller than p are gathered in the right and numbers that are equal to or larger than p are gathered in the left, no matter each block has how many numbers, the swap operation has the maximum number of times of execution, which equals to $\left\lfloor \frac{n}{2} \right\rfloor$. □

(d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is decremented).

Solution. No, it's not necessary to consider the worst-case input. With $n-1$ times of the while-loop execution and $\left\lfloor \frac{n}{2} \right\rfloor$ times of the swap execution for the worst-case input, the executed time is $\Omega(n)$; while with $n-1$ times of the while-loop execution and 0 times of the swap execution for the best-case input, the executed time is still $\Omega(n)$. Thus there's no need to consider the worst-case input.

□

(e) Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

Solution. The maximum total number of operations is when having a worst-case input, making it $n-1+\left\lfloor \frac{n}{2} \right\rfloor$. Then the upper bound of time complexity for the algorithm is $O(n)$. □

Question 4

Solve the following questions from the Discrete Math zyBook:

5.1.2 Consider the following definitions for sets of characters:

- Digits= $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Letters = $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$
- Special characters = $\{*, \&, \$, \#\}$

Compute the number of passwords that satisfy the given constraints.

(b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

Solution. The length of digits is 10, the length of letters is 26, the length of special characters is 4, total length is $10 + 26 + 4 = 40$. Thus, number of passwords that satisfy the given constraints is

$$40^7 + 40^8 + 40^9$$

□

(c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

Solution. The first character can be a digit or special character, the total length of digits and special characters is $10 + 4 = 14$. Thus the number of passwords that satisfy the given constraints is

$$14 \cdot (40^6 + 40^7 + 40^8)$$

□

5.3.2 Exercise 5.3.2, section a

(a) How many strings are there over the set $\{a, b, c\}$ that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

Solution.

$$3 \cdot 2^9 = 1536$$

□

5.3.3 License plate numbers in a certain state consists of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters (A through Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:
Digit-Letter-Letter-Letter-Letter-Digit-Digit

(b) How many license plate numbers are possible if no digit appears more than once?

Solution.

$$26^4 \cdot 10 \cdot 9 \cdot 8$$

□

(c) How many license plate numbers are possible if no digit or letter appears more than once?

Solution.

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 10 \cdot 9 \cdot 8$$

□

5.2.3 Let B $= \{0, 1\}$. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

(a) Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

Solution. For $f : B^9 \to E_{10}$, the function appends 1 or 0 to a nine-bit binary string. Whether to append 1 or 0 is being determined by the number of 1's in a nine-bit binary string. If a nine-bit binary string has an odd number of 1's, then it appends 1, otherwise it appends 0.

For $x_1 \in B^9, x_2 \in B^9$, if $x_1 \neq x_2$, as the first nine bits of the two strings are already different, what to append does not matter, then we have $f(x_1) \neq f(x_2)$, if $x_1 = x_2$, what to append to the string is already been determined by the string itself, then we have $f(x_1) = f(x_2)$, thus we say that f is one-to-one.

For any $y \in E_{10}$, as y is a ten-bit binary string, remove the last bit of y making it a nine-bit binary string x, by definition of B^9, $x \in B^9$ and $f(x) = y$, thus we can say that f is onto.

Therefore, $f : B^9 \to E_{10}$ is a bijection.

$\square$

(b) What is $|E_{10}|$?

Solution.

$$|E_{10}| = |B^9| = 2^9 = 512$$

$\square$

Question 5

Solve the following questions from the Discrete Math zyBook:

5.4.2 At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825.

(a) How many different phone numbers are possible?

Solution.

$$2 \cdot 10^4 = 20000$$

□

(b) How many different phone numbers are there in which the last four digits are all different?

Solution.

$$2 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 10080$$

□

5.5.3 How many 10-bit strings are there subject to each of the following restrictions?

(a) No restrictions.

Solution.

$$2^{10} = 1024$$

□

(b) The string starts with 001.

Solution.

$$2^7 = 128$$

□

(c) The string starts with 001 or 10.

Solution.

$$2^7 + 2^8 = 384$$

□

(d) The first two bits are the same as the last two bits.

Solution.

$$2^8 = 256$$

□

(e) The string has exactly six 0's.

Solution.

$$\binom{10}{4} = 210$$

□

(f) The string has exactly six 0's and the first bit is 1.

Solution.

$$\binom{9}{3} = 84$$

□

(g) There is exactly one 1 in the first half and exactly three 1's in the second half.

Solution.

$$\binom{5}{1}\binom{5}{3} = 50$$

□

5.5.5 Exercise 5.5.5, section a

(a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

Solution.

$$\binom{30}{10}\binom{35}{10}$$

□

5.5.8 This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

(c) How many five-card hands are made entirely of hearts and diamonds?

Solution.

$$\binom{26}{5} = 65780$$

□

(d) How many five-card hands have four cards of the same rank?

Solution.

$$\binom{13}{1}\binom{48}{1} = 624$$

□

(e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs. How many five-card hands contain a full house?

Solution.

$$\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3} = 3744$$

□

(f) How many five-card hands do not have any two cards of the same rank?

Solution.

$$\binom{13}{5} \cdot 4^5$$

□

5.6.6 A country has two political parties, the Demonstrators and the Repudiators. Suppose that the national senate consists of 100 members, 44 of which are Demonstrators and 56 of which are Repudiators.

(a) How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?

Solution.

$$\binom{44}{5}\binom{56}{5}$$

□

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

Solution.

$$P(44, 2) \cdot P(56, 2) = 44 \cdot 43 \cdot 56 \cdot 55 = 5827360$$

□

Question 6

Solve the following questions from the Discrete Math zyBook:

5.7.2 A 5-card hand is drawn from a deck of standard playing cards.

(a) How many 5-card hands have at least one club?

Solution.

$$\binom{52}{5} - \binom{39}{5}$$

□

(b) How many 5-card hands have at least two cards with the same rank?

Solution.

$$\binom{52}{5} - \binom{13}{5} \cdot 4^5$$

□

5.8.4 20 different comic books will be distributed to five kids.

(a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

Solution. 5^{20} □

(b) How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

Solution.

$$\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{20!}{4!4!4!4!4!}$$

□

Question 7

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4

Solution. 0
For $f : x \to y$, $|x| = 5! = 120$ and $|y| = 4! = 24$, $|y| < |x|$ thus f is not one-to-one. Therefore, there is no one-to-one function. □

b) 5

Solution. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ □

c) 6

Solution. $P(6, 5) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$ □

d) 7

Solution. $P(7, 5) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$ □