# Homework 11

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## Question 5

a. Use mathematical induction to prove that for any positive integer $n$, 3 divides $n^3 + 2n$ (leaving no remainder).

***Proof***. By induction on $n$.
**Base case:** $n = 1$. When $n = 1$, $1 + 2 = 3$, which can be divided by 3.
**Inductive step:** Assume that when $n = k$, $k^3 + 2k$ can be divided by 3, which means that there exists an integer $m$ such that $k^3+2k = 3m$.

Then for $k+1$, we have to prove that $(k+1)^3+2(k+1)$ can be divided by 3.

$$
\begin{aligned}
&(k+1)^3 + 2(k+1) \\
=&k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
=&(k^3 + 2k) + (3k^2 + 3k + 3) \\
=&3m + 3(k^2 + k + 1) \\
=&3(m + k^2 + k + 1)
\end{aligned}
$$

As $k$ and $m$ are both integers, $m + k^2 + k + 1$ is also an integer, thus $(k + 1)^3 + 2(k + 1)$ can be divided by 3.

Therefore, for any positive integer $n$, 3 divides $n^3 + 2n$.

□

b. Use strong induction to prove that any positive integer $n$ ($n \geq 2$) can be written as a product of primes.

***Proof.*** By induction on $n$.
**Base case:** $n = 2$. When $n = 2$, $2 = 1 \cdot 2$, 1 and 2 are prime numbers, thus it can be written as a product of primes.
**Inductive step:** Assume that for every integer between 2 and $k$, each can be written as a product of primes, which means that

$2 = 1 \cdot 2$
$3 = 1 \cdot 3$
$4 = 2 \cdot 2$
$\ldots$
$k = k_1 \cdot k_2 \cdots k_j$, where from $k_1$, $k_2$ till $k_j$ are all prime numbers

Then for $k+1$, if $k+1$ is a prime number, $k+1 = 1 \cdot (k+1)$;
if $k+1$ is not a prime number, let $a$ and $b$ be the two numbers that their product is $k+1$, which is $k+1 = a \cdot b$, where $a \geq 2$ and $b \geq 2$.

$$k+1 = a \cdot b$$
$$a = \frac{k+1}{b}$$
$$b = \frac{k+1}{a}$$

As $a \geq 2$ and $b \geq 2$,

$$\frac{k+1}{b} < k+1$$
$$\frac{k+1}{b} \leq k$$
$$a \leq k$$

$$\frac{k+1}{a} < k+1$$
$$\frac{k+1}{a} \leq k$$
$$b \leq k$$

As we already assumed that for every integer between 2 and $k$, each can be written as a product of primes, and $2 \leq a \leq k$ and $2 \leq b \leq k$, which means that

$a = a_1 \cdot a_2 \cdots a_m$, where from $a_1$, $a_2$ till $a_m$ are all primes.
$b = b_1 \cdot b_2 \cdots b_n$, where from $b_1$, $b_2$ till $b_n$ are all primes.

Then

$$\begin{aligned} k + 1 &= a \cdot b \\ &= (a_1 \cdot a_2 \cdots a_m)(b_1 \cdot b_2 \cdots b_m) \\ &= a_1 \cdot a_2 \cdot b_1 \cdot b_2 \cdots a_m \cdot b_n \end{aligned}$$

Thus we can see that for $k + 1$, no matter it's a prime number or not, it can be a product of primes.

Therefore, for any positive integer $n$ ($n \geq 2$) can be written as a product of primes.

$\square$

# Question 6

7.4.1 Define $P(n)$ to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that $P(3)$ is true.

*Solution.* When $n = 3$, the left of the equation is

$$\sum_{j=1}^{n} j^2 = 1^2 + 2^2 + 3^2 = 14$$

the right of the equation is

$$\frac{n(n+1)(2n+1)}{6} = \frac{3 \cdot 4 \cdot 7}{6} = 14$$

The left and right of the equation is equal, thus $P(3)$ is true. □

(b) Express $P(k)$.

*Solution.*

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

□

(c) Express $P(k+1)$.

*Solution.*

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

□

(d) In an inductive proof that for every positive integer $n$, what must be proven in the base case?

*Solution.* For an inductive proof, the base case is $P(1)$, thus $P(1)$ must be proved to be true. □

(e) In an inductive proof that for every positive integer $n$, what must be proven in the inductive step?

*Solution.* In the inductive step, we shall prove that

$$\text{For all } k \in \mathbb{Z}^+,\ P(k) \to P(k+1)$$

□

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

*Solution.* The inductive hypothesis is $P(k)$ is true. □

(g) Prove by induction that for any positive integer n, the assertion is true.

***Proof***. By induction on $n$.
**Base case:** $n = 1$. When $n = 1$,

$$\sum_{j=1}^{1} = 1^2 = 1$$

$$\frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6} = 1$$

The left and right of the equation is equal, thus $P(1)$ is true.

**Inductive step:** Assume that for positive integer $k$, $P(k)$ is true, there we have

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

For $k + 1$, the left of the equation is

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$$

the right of the equation is

$$\frac{(k+1)(k+2)(2k+3)}{6}$$

Given $P(k)$ is true, the difference between the left and right of the equation in $P(k+1)$ is

$$
\begin{aligned}
&1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 - \frac{(k+1)(k+2)(2k+3)}{6} \\
=&\frac{k(k+1)(2k+1)}{6} + (k+1)^2 - \frac{(k+1)(k+2)(2k+3)}{6} \\
=&(k+1)(\frac{2k^2+k}{6} + (k+1) - \frac{2k^2+7k+6}{6}) \\
=&(k+1)(k+1-\frac{6k+6}{6}) \\
=&0
\end{aligned}
$$

Thus the difference between the left and right of the equation is 0, which means that $P(k+1)$ is true.

Therefore, for any positive integer n, the assertion is true.

□

7.4.3 (c) Prove that for $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

***Proof***. By induction on $j$.
Let $P(n)$ be the assertion that

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

**Base case:** $n = 1$. When $n = 1$, the left of $P(1)$ is

$$\frac{1}{1^2} = 1$$

the right of $P(1)$ is

$$2 - \frac{1}{1} = 1$$

As $1 \leq 1$, $P(1)$ is true.

**Inductive step:** Assume that when $n = k$, the assertion $P(k)$ is true, there we have

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

When $n = k + 1$,

$$\begin{aligned} \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\ &\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} \\ &= 2 - \frac{1}{k+1} \end{aligned}$$

Thus, $P(k + 1)$ is true.

Therefore for $n \geq 1$ the assertion is true.

□

7.5.1 (a) Prove that for any positive integer $n$, 4 evenly divides $3^{2n} - 1$.

***Proof***. By induction on $n$.
Let $f(n) = 3^{2n} - 1$.
**Base case:** $n = 1$. When $n = 1$, $f(1) = 3^2 - 1 = 8$, which can be evenly divided by 4, as the remainder is 0.
**Inductive step:** Assume that when $n = k$, $f(k) = 3^{2k} - 1$ can be evenly divided by 4, which means that there exists an integer $m$ such that $f(n) = 4m$.
Then, when $n = k + 1$

$$\begin{aligned} f(k+1) &= 3^{2(k+1)} - 1 \\ &= 9 \cdot 3^{2k} - 9 + 8 \\ &= 9 \cdot (3^{2k} - 1) + 2 \cdot 4 \\ &= 9 \cdot f(k) + 2 \cdot 4 \\ &= 9 \cdot 4m + 2 \cdot 4 \\ &= 4(9m + 2) \end{aligned}$$

As $m$ is an integer, then $9m + 2$ is also an integer, which means that $f(k + 1)$ can be evenly divided by 4.

Therefore, for any positive integer $n$, 4 evenly divides $3^{2n} - 1$.

□